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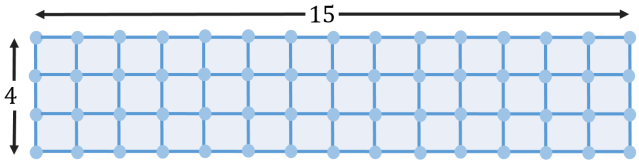
# Algorithm Theory - Winter Term 2017/2018
# Exercise Sheet 2

**Hand in by Thursday 10:15, November 16, 2017**

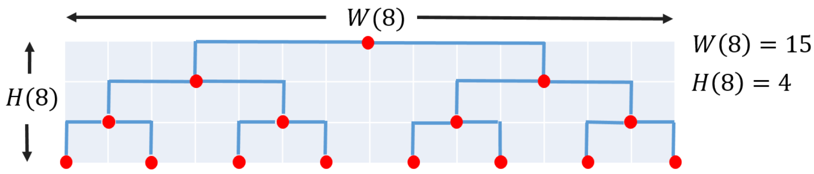
## Exercise 1: Tree Embedding into Grids (4+6 Points)

A $n \times m$ grid graph is a graph $G = (V_G, E_G)$ with nodes $V_G := \{(i, j) \mid i \in \{1, \dots, n\}, j \in \{1, \dots, m\}\}$. These nodes are embedded in the Euclidian plane $\mathbb{R}^2$ and connected with edges as exemplified by the following $4 \times 15$ grid graph.




An embedding of a tree $T = (V_T, E_T)$ into a grid graph $G = (V_G, E_G)$ is defined as a one to one mapping of $V_T$ to a subset of $V_G$, which satisfies the following condition. There exists a set $P_G$ of vertex-disjoint paths[1] in $G$ such that for each $\{u, v\} \in E_T$, there is a path $p \in P_G$ connecting $u'$ to $v'$, when $u$ is mapped to $u'$, and $v$ is mapped to $v'$.

(a) We can embed a complete binary tree with $n$ leaves into a grid, such that the nodes with height $i$ of the tree are placed in the $i^{th}$ row of the grid. Below you see the embedding of a tree with 8 leaves into a $4 \times 15$ grid as an example.




By this way of embedding, show that we need a grid of size[2] $\Theta(n \log n)$ to embed a complete tree with $n$ leaves. To do so, write down the recurrence relations for the width $W(n)$ and the height $H(n)$ of the grid.

(b) Find a more efficient way of embedding a complete binary tree and show that it needs a grid of size $\Theta(n)$, if the tree has $n$ leaves. Write down the recurrence relations for the width $W(n)$ and the height $H(n)$ of the grid.

[1] We define vertex-disjoint paths in $G$ as paths that may only have common endpoints but are disjoint otherwise.
[2] The size of a $n \times m$ grid graph is simply $n \cdot m$.

## Exercise 2: Polynomial to the power of $k$ (4+6 Points)

Given a polynomial $p(x)$ of degree $n$ and an integer $k \geq 2$, the goal of this problem is to compute the $k^{th}$ power $p^k(x)$ of $p(x)$ in an efficient way. For simplicity, we assume that $k$ is a power of 2, that is, $k = 2^\ell$ for some integer $\ell \geq 1$.

(a) Describe an efficient algorithm to compute $p^k(x)$ polynomial using the *Fast Polynomial Multiplication* algorithm from the lecture.

(b) What is the asymptotic runtime of your algorithm in terms of $k$ and $n$? Explain your answer.

## Exercise 3: Greedy Algorithm (10 Points)

In the following, a *unit fraction* is a fraction where the numerator is 1 and the denominator is some integer larger than 1. For example 1/4 or 1/384 are unit fractions.

It is well-known that every rational number $0 < q < 1$ can be expressed as a sum of pairwise distinct unit fractions, e.g., we can write $\frac{4}{13}$ as

$$\frac{4}{13} = \frac{1}{5} + \frac{1}{13} + \frac{1}{32} + \frac{1}{65}.$$

Interestingly such a decomposition into distinct unit fractions can be computed using a simple greedy algorithm.

In the following, assume that you are given two positive integers $a$ and $b$ such that $b > a$. Design a greedy algorithm to compute integers $0 < c_1 < c_2 < \cdots < c_k$ such that

$$\frac{a}{b} = \frac{1}{c_1} + \frac{1}{c_2} + \cdots + \frac{1}{c_k}.$$

Prove that your greedy algorithm always works and that it decomposes $\frac{a}{b}$ into at most $a$ unit fractions.

You can assume that your algorithm can deal with arbitrarily large integer numbers. Note that for the fraction $\frac{4}{13}$, the standard greedy algorithm computes a decomposition which is different from the one given above.

## Exercise 4: Matroids (6+4 Points)

(a) For a graph $G = (V, E)$, a subset $F \subseteq E$ of the edges is called a forest iff (if and only if) it does not contain a cycle. Let $\mathcal{F}$ be the set of all forests of $G$. Show that $(E, \mathcal{F})$ is a matroid.

*Hint: A forest with $k$ edges and $n$ nodes has $n - k$ connected components.*

(b) For a matroid $(E, I)$, a maximal independent set $S \in I$ is an independent set that cannot be extended. Thus, for every element $e \in E \setminus S$, the set $S \cup \{e\} \notin I$.

What are the maximal independent sets of the matroid in (a)?